Mirror symmetry for toric branes on compact hypersurfaces

JHEP09(2009)126

(http://iopscience.iop.org/1126-6708/2009/09/126)

Published by IOP Publishing for SISSA

Received: *August 13, 2009*
Accepted: *August 25, 2009*
Published: *September 29, 2009*

Mirror symmetry for toric branes on compact hypersurfaces

M. Alim, M. Hecht, P. Mayr and A. Mertens

Arnold Sommerfeld Center for Theoretical Physics, LMU,
Theresienstr. 37, D-80333 Munich, Germany
E-mail: murad.alim@physik.uni-muenchen.de, michael.hecht@physik.uni-muenchen.de, mayr@physik.uni-muenchen.de, adrian.mertens@physik.uni-muenchen.de

Abstract: We use toric geometry to study open string mirror symmetry on compact Calabi-Yau manifolds. For a mirror pair of toric branes on a mirror pair of toric hypersurfaces we derive a canonical hypergeometric system of differential equations, whose solutions determine the open/closed string mirror maps and the partition functions for spheres and discs. We define a linear sigma model for the brane geometry and describe a correspondence between dual toric polyhedra and toric brane geometries. The method is applied to study examples with obstructed and classically unobstructed brane moduli at various points in the deformation space. Computing the instanton expansion at large volume in the flat coordinates on the open/closed deformation space we obtain predictions for enumerative invariants.

Keywords: D-branes, Topological Strings

ArXiv ePrint: 0901.2937

doi:10.1088/1126-6708/2009/09/126

Contents

1 Introduction

Mirror symmetry has been the subject of intense research over many years and its study remains rewarding. Whereas the early works focused on the closed string sector and the Calabi-Yau (CY) geometry, the interest has shifted to the interpretation of mirror symmetry as a duality of D-brane categories and the associated open string sector [1]. One object of particular interest is the disc partition function $\mathcal{F}^{0,1}$ for an A brane on a compact CY

3-fold, which depends on the Kähler type deformations of the brane geometry and is an important datum for the definition of the category of A branes. If a modulus is classically unobstructed, the large volume expansion of the disc partition function captures an interesting enumerative problem of "counting" holomorphic discs that end on the A brane. In a certain parametrization motivated by physics, the coefficients of this instanton expansion in the A model are predicted to be the *integral* Ooguri-Vafa invariants [2].

One of the virtues of mirror symmetry, first demonstrated for the sphere partition function in [3] and for the disc partition functions in [4], is the ability to compute the instanton expansion of the A model partition function in the mirror B model. The disc partition function relates on the B model side to the holomorphic Chern-Simons functional on the CY Z^* [5]

$$S(Z^*, A) = \int_{Z^*} \mathrm{tr}\left(\frac{1}{2}\mathrm{A} \wedge \mathrm{dA} + \frac{1}{3}\mathrm{A} \wedge \mathrm{A} \wedge \mathrm{A}\right) \wedge \Omega. \tag{1.1}$$

In the physical string theory S represents a space-time superpotential obstructing some of the moduli of the brane geometry and the instanton expansion of the A model is, under certain conditions, the non-perturbative superpotential generated by space-time instantons [6, 2]. While the action of mirror symmetry on the moduli space and the computation of superpotentials is well understood for non-compact brane geometries,[1] the physically interesting case of branes on compact CY 3-folds has been elusive. Starting with [8], superpotentials for a class of involution branes without open string moduli have been studied in [9–13]. The definition of the Lagrangian A brane geometry as the fixed point of an involution has various limitations: It allows to study only discrete brane moduli compatible with the involution and the instanton invariants computed by the superpotential are not generic disc invariants, but rather the number of real rational curves fixed by the involution [14].

The present lack of a systematic description of the geometric deformation space in the compact case is a serious obstacle to the general study of open string mirror symmetry on compact manifolds, in particular the computation of superpotentials and mirror maps for more general deformations including open string moduli. For the closed string case without branes, a powerful approach to study mirror symmetry is given in terms of gauged linear sigma models and toric geometry [15, 16], in particular if combined with Batyrev's construction of dual manifolds via toric polyhedra [17].[2] A similar description of open string mirror symmetry has been given for non-compact branes in [19, 20], starting from the definition of toric branes of ref. [4]. A first important step to generalize these concepts to the compact case has been made in [13] by applying the $\mathcal{N} = 1$ special geometry defined in [20] to involution branes.

The class of toric branes defined in [4] (see also [21]) is much larger then the class of involution branes and allows for relatively generic deformations. The purpose of this note is to describe a toric geometry approach to open string mirror symmetry for toric branes on compact manifolds. Specifically we consider mirror pairs (Z, L) and (Z^*, E),

[1] See e.g. [7] for a summary.

[2] We refer to [18] for background material and references.

where Z and Z^* is a mirror pair of compact CY 3-folds described as hypersurfaces in toric varieties, and L and E is a mirror pair of branes on these manifolds with a simple toric description.[3] For these toric brane geometries we derive in section 2 a canonical system of differential equations that determines the open/closed string mirror maps and the partition functions for spheres and discs at any point in the moduli space. The B model geometry for this Picard-Fuchs system relates to a certain gauged linear sigma model, which may be associated with an "enhanced" toric polyhedron $\Delta_\flat$. A dual pair of enhanced polyhedra $(\Delta_\flat, \Delta_\flat^\star)$ encodes the mirror pair of compact CY manifolds (Z, Z^*) *and* the mirror pair (L, E) of A and B branes on it, extending in some sense Batyrev's [17] correspondence between toric polyhedra and CY manifolds to the open string sector. In section 3 we apply this method to study some compact toric brane geometries with obstructed and classically unobstructed moduli. The phase structure of the linear sigma model can be used to define and study large volume phases of the brane geometry, where the superpotential has an instanton expansion in the classically unobstructed moduli. We compute the mirror maps and the superpotentials and find agreement with the integrality predictions of [2, 8] for both closed and open string deformations. A more complete treatment and derivations of some of the formula presented below are deferred to an upcoming paper [22].

2 Toric brane geometries and differential equations

2.1 Toric hypersurfaces and branes

Our starting point will be a mirror pair of compact CY 3-folds (Z, Z^*) defined as hypersurfaces in toric varieties (W, W^*). By the correspondence of ref. [17], one may associate to the pair of manifolds (Z, Z^*) a pair of integral polyhedra (Δ, Δ^*) in a four-dimensional integral lattice Λ_4 and its dual Λ_4^*. The k integral points $\nu_i(\Delta)$ of the polyhedron Δ correspond to homogeneous coordinates x_i on the toric ambient space W and satisfy $M = h^{1,1}(Z)$ linear relations[4]

$$\sum_i l_i^a \nu_i = 0, \quad a = 1, \dots, M\,.$$

The integral entries of the vectors l^a for fixed a define the weights l_i^a of the coordinates x_i under the $\mathbf{C}^*$ action

$$x_i \to (\lambda_a)^{l_i^a} x_i, \qquad \lambda_a \in \mathbf{C}^*\,,$$

generalizing the idea of a weighted projective space. Equivalently, the l_i^a are the $\mathrm{U}(1)_a$ charges of the fields in the gauged linear sigma model (GLSM) associated with the toric variety [15]. The toric variety W is defined as $\mathbf{C}^k$ divided by the $(\mathbf{C}^*)^M$ action and deleting a certain exceptional subset Ξ of degenerate orbits, $W \simeq (\mathbf{C}^k - \Xi)/(\mathbf{C}^*)^M$.

In the context of CY hypersurfaces, W will be the total space of the anti-canonical bundle over a toric variety with positive first Chern class. The compact manifold $Z \subset W$ is defined by introducing a superpotential $W_Z = x_0\, p(x_i)$ in the GLSM, where x_0 is the

[3] In the following, L will denote the A brane wrapped on a Lagrangian submanifold and E the holomorphic bundle corresponding to a B brane.

[4] For simplicity we neglect points on faces of codimension one of Δ and assume that $h^{1,1}(W) = h^{1,1}(Z)$.

coordinate on the fiber and $p(x_i)$ a polynomial in the $x_{i>0}$ of degrees $-l_0^a$. At large Kähler volumes, the critical locus is at $x_0 = p(x_i) = 0$ and defines the compact CY as the hypersurface $Z:\ p(x_i) = 0$ [15]. To be concrete, we will later study A branes on the following examples of CY hypersurfaces:

$$\begin{array}{ll}
\mathbf{X}_5^{(1,1,1,1,1)} & \begin{array}{cc} & \begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ (l^1) = & \left(\begin{array}{cccccc} -5 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array} \\
\mathbf{X}_{18}^{(1,1,1,6,9)} & \begin{array}{cc} & \begin{array}{ccccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ (l^1) = & \left(\begin{array}{ccccccc} -6 & 3 & 2 & 1 & 0 & 0 & 0 \end{array}\right) \\ (l^2) = & \left(\begin{array}{ccccccc} 0 & 0 & 0 & -3 & 1 & 1 & 1 \end{array}\right) \end{array} \\
\mathbf{X}_9^{(1,1,1,3,3)} & \begin{array}{cc} & \begin{array}{ccccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ (l^1) = & \left(\begin{array}{ccccccc} -3 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \\ (l^2) = & \left(\begin{array}{ccccccc} 0 & 0 & 0 & -3 & 1 & 1 & 1 \end{array}\right) \end{array}
\end{array} \tag{2.1}$$

As indicated by the notation, this is the familiar quintic in projective space $\mathbf{P}^4 = \mathbf{WP}^4_{1,1,1,1,1}$ in the first case and a degree 18 (9) hypersurface in a blow up of a weighted projective space $\mathbf{WP}^4_{1,1,1,6,9}$ ($\mathbf{WP}^4_{1,1,1,3,3}$) in the other two cases.[5]

On these toric manifolds we consider a certain class of mirror pairs of branes, defined in [4] by another set of N charge vectors $\hat{l}^a$ for the fields x_i.[6] The Lagrangian submanifold wrapped by the A brane L is described in terms of the vectors $\hat{l}^a$ by the equations

$$\sum_i \hat{l}_i^a |x_i|^2 = c_a, \qquad \sum_i v_b^i \theta^i = 0, \qquad \sum_i \hat{l}_i^a v_b^i = 0\,, \tag{2.2}$$

where $a, b = M+1, \ldots, M+N$. The N real constants c_a parametrize the brane position and the integral vectors v_b^i may be defined as a linearly independent basis of solutions to the last equation. As in [4] we restrict to special Lagrangians which requires that the extra charges add up to zero as well, $\sum_i \hat{l}_i^a = 0$.

Applying mirror symmetry as in [23, 17], the mirror manifold Z^* is defined in the toric variety W^* by the equations

$$p(Z^*) = \sum_i y_i, \qquad \prod_i y_i^{l_i^a} = z_a\,, \quad a = 1, \ldots, M. \tag{2.3}$$

The parameters z_a are the complex moduli of the hypersurface Z^* and classically related to the complexified Kähler moduli t_a of Z by $z_a = e^{2\pi i t_a}$. The precise relation $z_a = z_a(t_b)$ is called the mirror map and is generically complicated. In the open string sector, the mirror transformation of [23] maps the A brane (2.2) to a B brane E defined by the holomorphic equations [4]

$$\mathcal{B}_a(E): \prod_i y_i^{\hat{l}_i^a} - \hat{z}_a = 0, \qquad \hat{z}_a = \epsilon_a e^{-c_a}, \ a = M+1, \ldots, M+N. \tag{2.4}$$

[5]The deleted set is $\Xi = \{x_i = 0, \forall i > 0\}$ for $\mathbf{P}^4$ and $\Xi = \{\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}\}$ in the other two cases. The toric polyhedra will be given in section 3.

[6]A hat will be sometimes used to distinguish objects from the open string sector.

JHEP09(2009)126

The (possibly obstructed) complex open string moduli $\hat{z}_a$ arise from the combination of the phases ϵ_a dual to the gauge field background on the A brane and the parameters c_a in (2.2) [24].

The class of toric branes defined above is quite general and describes many interesting cases, in particular involution branes with an obstructed modulus as well as branes with classically unobstructed moduli. It is instructive to consider the quintic $\mathbf{X}_5^{(1,1,1,1,1)}$, which will be one of the manifolds studied in section 3. The manifold Z for the A model is defined by a generic degree 5 polynomial in $\mathbf{P}^4$, while the mirror manifold Z^* is given in terms of eq. (2.3) by the superpotential and relation[7]

$$p(Z^*) = \sum_{i=0}^{5} a_i y_i = 0, \qquad y_1 y_2 y_3 y_4 y_5 = y_0^5. \tag{2.5}$$

A change of coordinates $y_i = x_i^5,\ i = 1,\dots,5$ and a rescaling leads to the more familiar form in $\mathbf{P}^4$

$$p(Z^*) = \sum_{i=1}^{5} x_i^5 - \psi\, x_1 x_2 x_3 x_4 x_5 = 0, \qquad \psi^{-5} = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5} \equiv z_1\ . \tag{2.6}$$

The above definition of toric branes has an interesting overlap with more recent studies of B branes via matrix factorizations.[8] Consider the charge vectors

$$\begin{array}{|cc|cccccc|}\hline & & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline l^1 & = & -5 & 1 & 1 & 1 & 1 & 1 \\ \hat{l}^2 & = & 0 & 1 & -1 & 0 & 0 & 0 \\ \hat{l}^3 & = & 0 & 0 & 0 & 1 & -1 & 0 \\ \hline \end{array} \tag{2.7}$$

For the special values $c_a = 0$ the equation (2.2) for the Lagrangian submanifold can be rewritten as

$$x_1 = \bar{x}_2,\ \ x_3 = \bar{x}_4,\ \ x_5 = \bar{x}_5\ .$$

The above equation describes an involution brane on the quintic defined as the fixed set of the $\mathbf{Z}_2$ action $(x_1, x_2, x_3, x_4, x_5) \to (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3, \bar{x}_5)$. The equation for the mirror B-brane follows from (2.4):

$$y_1 = \hat{z}_2\, y_2, \quad y_3 = \hat{z}_3\, y_4, \qquad \text{or} \qquad x_1^5 = \hat{z}_2\, x_2^5, \quad x_3^5 = \hat{z}_3\, x_4^5\ . \tag{2.8}$$

A naive match of the moduli of the A and B model together with a choice of phase leads to $\hat{z}_2 = \hat{z}_3 = -1$ and the above equations become

$$x_1^5 + x_2^5 = 0,\ \ x_3^5 + x_4^5 = 0,\ \ (x_5^2 - \psi^{1/2} x_1 x_3)\,(x_5^2 + \psi^{1/2} x_1 x_3)\, x_5 = 0\ . \tag{2.9}$$

These equations define a set of holomorphic 2-cycles in Z^* which may be wrapped by the D5 brane mirror to the A brane on the Lagrangian subset defined by (2.2).

[7]The coefficients a_i are homogeneous coordinates on the space of complex structure and related to the z_a in (2.3) by an rescaling of the variables y_i.

[8]See refs. [25] for a summary.

Eq. (2.9) should be compared to the results of refs. [8, 9], where the 2-cycle wrapped by the B brane mirror to an involution brane has been determined in a much more involved way along the lines of [26], by proposing a matrix factorization and computing the second algebraic Chern class of the associated complex. The result agrees with the above result from a simple application of mirror symmetry for toric branes. A conclusive match of the toric brane defined by (2.4) and the matrix factorization brane studied in [8, 9] will given in section 3, where we compute the superpotential from the toric family and find agreement near a specific critical locus.

There are ambiguities in the above match between the A and the B model that need to be resolved by a careful study of boundary conditions. e.g. in (2.9), the last equation is the superpotential intersected with the two hypersurfaces (2.8), but one may permute the meaning of the three equations. The parametrization

	x_0	x_1	x_2	x_3	x_4	x_5
$l^1 =$	-5	1	1	1	1	1
$\hat{l}^2 =$	1	0	0	0	0	-1
$\hat{l}^3 =$	0	1	-1	0	0	0

(2.10)

leads to the same equations (2.9) for the special values $\hat{z}_2 = \psi$, $\hat{z}_3 = -1$ of the (new) moduli. An important aspect in resolving these ambiguities is provided by the mirror map $z_a(t_b)$ on the open/closed moduli space, as it determines where a specific (family of) point(s) in the A moduli attaches in the moduli space of the B model and vice versa. In the above example we have simply used the classical version of the open string mirror maps $|\hat{z}_a| = e^{-c_a}$ to find agreement with the result from matrix factorizations. More seriously we will compute the exact mirror map – which may in principle deviate substantially from the classical expression – to determine the B brane configuration. Since some of the deformations will be fixed at the critical points of the superpotential it is in fact more natural to start with the computation of the B model superpotentials and to find its critical points. Computing the mirror map near these points determines a correlated set of points in the A model parameter space, which may or may not allow for a nice classical A brane interpretation.

2.2 $\mathcal{N} = 1$ special geometry of the open/closed deformation space

We proceed by discussing a general structure of the open/closed deformation space that will be central to the following approach to mirror symmetry for the toric branes defined above. In [20, 13] it was shown, that the open/closed string deformation space for B-type D5-branes wrapping 2-cycles C in Z^* can be studied from the variation of mixed Hodge structure on a deformation family of relative cohomology groups $H^3(Z^*, \mathcal{H})$ of Z^*, where $\mathcal{H}$ is a subset that captures the deformations of C.[9] In the simplest case, $\mathcal{H}$ is a single

[9] Physically, $\mathcal{H}$ may be interpreted as a D7-brane which contains the D5 brane world-volume [22].

hypersurface and the action of the closed and open string variations is schematically

$$\begin{tikzcd}
F^3H^3(Z^*) \arrow[r, "\delta_z"] \arrow[dr, "\delta_{\hat z}"] & F^2H^3(Z^*) \arrow[r, "\delta_z"] \arrow[dr, "\delta_{\hat z}"] & F^1H^3(Z^*) \arrow[r, "\delta_z"] \arrow[dr, "\delta_{\hat z}"] & F^0H^3(Z^*) \arrow[d, "\delta_{\hat z}"] \\
& F^2H^2(\mathcal{H}) \arrow[r, "\delta_z, \delta_{\hat z}"] & F^1H^2(\mathcal{H}) \arrow[r, "\delta_z, \delta_{\hat z}"] & F^0H^2(\mathcal{H})
\end{tikzcd} \tag{2.11}$$

Here F^k is the Hodge filtration and δ_z and $\delta_{\hat{z}}$ denote the closed and open string variations, respectively. For more details on this structure we refer to refs. [20, 13] (see also [27, 28]). The variations δ can be identified with the flat Gauss-Manin connection ∇, which captures the variation of mixed Hodge structure on the bundle with fibers the relative cohomology groups. The mathematical background is described in refs. [29–31].

The flatness of the Gauss-Manin connection leads to a non-trivial "$\mathcal{N} = 1$ special geometry" of the combined open/closed field space, that governs the open/closed chiral ring of the topological string theory [20]. This geometric structure leads to a Picard-Fuchs system of differential equations satisfied by the relative period integrals

$$\mathcal{L}_a \Pi_\Sigma = 0, \qquad \Pi_\Sigma(z, \hat{z}) = \int_{\gamma_\Sigma} \Omega, \qquad \gamma_\Sigma \in H_3(Z^*, \mathcal{H}) \ . \tag{2.12}$$

Here $\{\mathcal{L}_a\}$ is a system of linear differential operators, $z(\hat{z})$ stands collectively for the closed (open) string parameters and the holomorphic 3-form Ω and its period integrals are defined in relative cohomology. The relative periods $\Pi_\Sigma(z, \hat{z})$ determine the mirror map and the combined open/closed string superpotential, which can be written in a unified way as

$$\mathcal{W}_{\mathcal{N}=1}(z, \hat{z}) = \mathcal{W}_{\text{closed}}(z) + \mathcal{W}_{\text{open}}(z, \hat{z}) = \sum_{\gamma_\Sigma \in H^3(Z^*, \mathcal{H})} N_\Sigma \Pi_\Sigma(z, \hat{z}) \ . \tag{2.13}$$

Here $\mathcal{W}_{\text{closed}}(z)$ is the closed string superpotential proportional to the periods over cycles $\gamma_\Sigma \in H^3(Z^*)$ and $\mathcal{W}_{\text{open}}(z, \hat{z})$ the brane superpotential proportional to periods over chains γ_Σ with non-empty boundary $\partial\gamma_\Sigma$. The coefficients N_Σ are the corresponding "flux" and brane numbers.[10]

In the following we implement this general structure for the class of toric branes on compact manifolds defined in section 2.1. In the present context, the deformations of C are controlled by eq. (2.4) and the relative cohomology problem is naturally defined by the hypersurfaces $\mathcal{B}_a(E)$ in the B model. In [20] this identification was used to set up the appropriate problem of mixed Hodge structure for branes in non-compact CY manifolds and to compute the Picard-Fuchs system of the $\mathcal{N} = 1$ special geometry. This approach was extended to the compact case in [13] by relating $\mathcal{H}$ to the algebraic Chern class $c_2(E)$ of a B brane as obtained from a matrix factorization. As observed in section 2, these two definitions of $\mathcal{H}$ are closely related and it is straightforward to check that they coincide in

[10] To obtain the physical superpotential, an appropriate choice of reference brane has to be made for the chain integrals, since a relative period more precisely computes the brane tension of a domain wall [32, 33, 4]. This should be kept in mind in the following discussion where we simply refer to "the superpotential".

concrete examples; in particular the hypersurfaces defined in [13] fit into the definition of $\mathcal{H}$ via (2.4) in [20].[11]

2.3 GLSM and enhanced toric polyhedra

To make full use of the machinery of toric geometry we start with defining a GLSM for the CY/brane geometry. The GLSM puts the CY geometry and the brane geometry on equal footing and allows to study the phases of the combined system by standard methods of toric geometry . The GLSM thus provides valuable information on the *global* structure of the combined open/closed deformation space which will be important for identifying and investigating the various phases of the brane geometry, in particular large volume phases.

We will use the concept of toric polyhedra to define the GLSM for the mirror pairs of toric brane geometries. This approach has the advantage of giving a canonical construction of the B model mirror to a certain A brane geometry and provides a short-cut to derive the generalized hypergeometric system for the relative periods given in eq.(2.18) below. As discussed above, Batyrev's correspondence describes a mirror pair of toric hypersurfaces (Z, Z^*) by a pair of dual polyhedra $(\Delta, \Delta^\star)$. What we are proposing here is that there is a similar correspondence between "enhanced polyhedra" $(\Delta_\flat(Z, L), \Delta_\flat^\star(Z^*, E))$ and the pair (Z, Z^*) of mirror manifolds *together* with the pair of mirror branes (L, E) as defined before.

The enhanced polyhedron $\Delta_\flat(Z, L)$ has the following simple structure: The points $\nu_i(Z)$ of $\Delta(Z)$ defining the manifold Z are a subset of the points of $\Delta_\flat(Z, L)$ that lie on a hypersurface H in a five-dimensional lattice Λ_5. We choose an ordering of the points $\mu_i \in \Delta_\flat(Z, L)$ and coordinates on Λ_5 such that the points in H are given by

$$(\mu_i) = (\nu_i, 0),\ \ i = 1, \ldots, k\,,$$

where k is the number of points of $\Delta(Z)$. The brane geometry is described by k' extra points ρ_i with $(\rho_i)_5 < 0$, where k' is related to the number $\hat{n}$ of (obstructed) moduli of the brane by $k' = \hat{n} + 1$. Thus $\Delta_\flat(Z, L)$ is defined as the convex hull of the points

$$\Delta_\flat(Z, L) = \mathrm{conv}\left(\{\mu_i(\Delta(Z))\} \cup \{\rho_i(L)\}\right), \qquad \{\mu_i(\Delta(Z))\} \subset \Delta_\flat(Z, L) \cap H\,, \tag{2.14}$$

For simplicity we assume that the polyhedron $\Delta_\flat^\star$ can be naively defined as the dual of $\Delta_\flat$ in the sense of [17].

To make contact between the definition of the toric branes in section 2 and the extra points ρ_i, consider the linear dependences between the points of $\Delta_\flat(Z, L)$

$$\sum_i \underline{l}_i^a(\Delta_\flat)\mu_i = 0\ . \tag{2.15}$$

These relations may be split into two sets in an obvious way. There are $h^{1,1}(Z)$ relations, say

$$(\underline{l}^a(\Delta_\flat)) = (l^a(\Delta), 0^{k'}),\ \ a = 1, \ldots, h^{1,1}(Z)\,,$$

[11] As was stressed in section 3.6 of [9], the chain integrals, which define the *normal functions* associated with the superpotential, do not depend on the details of the infinite complexes constructed in [26]. Our results suggest that the relevant information for the superpotential is captured by the linear sigma model defined below.

JHEP09(2009)126

which involve only the first k points and reflect the original relations $l^a(\Delta)$ between the points $\nu_i(Z)$ of $\Delta(Z)$; they correspond to Kähler classes of the manifold Z. The remaining relations $\underline{l}^a(\Delta_\flat)$, $a > h^{1,1}(Z)$ involve also the extra points ρ_i. To describe a brane as defined by the charge vectors $\hat{l}^a(L)$ we choose the points ρ_i such that the remaining relations are of the form

$$(\underline{l}^a(\Delta_\flat)) = (\hat{l}^a(L), \dots), \ a > h^{1,1}(Z) \ .$$

The above prescription for the construction of the enhanced polyhedron $\Delta_\flat(Z, L)$ from the polyhedron $\Delta(Z)$ for a given manifold Z and the definition (2.2) of the A brane L in section 2 is well-defined if we require a minimal extension by $k' = \hat{n} + 1$ points.

2.4 Differential equations on the moduli space

The combined open/closed string deformation space of the brane geometries (Z, L) or (Z^*, E) can now be studied by standard methods of toric geometry. Let[12] $\{l^a_i\}$ denote a *specific* choice of basis for the generators of the relations (2.15) in the GLSM and a_i the coefficients of the hypersurface equation $p = \sum_i a_i y_i$ of the mirror B model. From the homogeneous coordinates a_i on the complex moduli space one may define local coordinates associated with the choice of a basis $\{l^a_i\}$ by[13]

$$z_a = (-)^{l^a_0} \prod_i a_i^{l^a_i} \ , \qquad a = 1, \dots, M + N. \tag{2.16}$$

Our main tool will be a system of linear differential equations of the form

$$\mathcal{L}_a \Pi(z_b) = 0, \tag{2.17}$$

whose solutions are the relative periods (2.12). The relative periods determine not only the genus zero partition functions but also the mirror map $z_a(t_b)$ between the flat coordinates t_a and the algebraic moduli z_a for the open/closed string deformation space [20]. There are two ways to derive the system of differential operators $\{\mathcal{L}_a\}$: Either as the canonical generalized hypergeometric GKZ system associated with the enhanced polyhedron $\Delta_\flat(Z, L)$ [35, 17]. Or as the system of differential equations capturing the variation of mixed Hodge structure on the relative cohomology group $H^3(Z^*, \mathcal{H})$ as in refs. [20, 13].

Here we use the short-cut of toric polyhedra and define the Picard-Fuchs system as the canonical GKZ system associated with $\Delta_\flat$.[14] The derivation of the Picard-Fuchs system from the variation of mixed Hodge structure on the relative cohomology group, which is similar to that in [20], will be given in [22]; the coincidence of the two definitions is non-trivial and reflects a string duality [36, 22]. By the results of [35, 17], the generalized hypergeometric system associated to $(\Delta_\flat, \Delta^\star_\flat)$ leads to the following differential operators

[12] The underscore on $\underline{l}^a(\Delta_\flat)$ will be dropped again to simplify notation.

[13] The sign is a priori convention but receives a meaning if the classical limit of the mirror map is fixed as in [34].

[14] We are tacitly assuming that the GKZ system $\{\mathcal{L}_a\}$ is already a complete Picard-Fuchs system, which is possibly only true after a slight modification of the GKZ system.

JHEP09(2009)126

for $a = 1, \dots, M+N$:

$$\mathcal{L}_a = \prod_{k=1}^{l_0^a} (\theta_{a_0} - k) \prod_{l_i^a > 0} \prod_{k=0}^{l_i^a - 1} (\theta_{a_i} - k) - (-1)^{l_0^a} z_a \prod_{k=1}^{-l_0^a} (\theta_{a_0} - k) \prod_{l_i^a < 0} \prod_{k=0}^{-l_i^a - 1} (\theta_{a_i} - k) \tag{2.18}$$

Here θ_x denotes a logarithmic derivative $\theta_x = x\frac{\partial}{\partial x}$ and the derivatives of the homogeneous coordinates a_i on the complex structure moduli and the local coordinates (2.16) are related by $\theta_{a_i} = \sum_a l_i^a \theta_{z_a}$. The products are defined to run over non-negative k only so that the derivatives θ_{a_0} appear only in one of the two terms for given a. The solutions of the Picard-Fuchs system in eq. (2.18) have a nice expansion around $z_a = 0$; expansions around other points in the moduli space can be obtained from a change of variables.

Eqs. (2.17), (2.18) represent the homogeneous Picard-Fuchs system for the brane geometry (Z^*, E). These homogeneous Picard-Fuchs equations give rise to inhomogeneous Picard-Fuchs equations by splitting the operators $\mathcal{L}_a$ in a piece $\mathcal{L}_{a,bulk}$ that depends only on the moduli z of the manifold Z^* and essentially represent the Picard-Fuchs system of the CY geometry and a part $\mathcal{L}_{a,open}$ that governs the dependence on the open string deformations $\hat{z}$. Upon evaluation at a critical point w.r.t. the open string deformations, $\delta_{\hat{z}} \mathcal{W} = 0$, the split leads to an inhomogeneous term $f_a(z)$, if Π is a chain that depends non-trivially on the brane deformations $\hat{z}$.

$$\mathcal{L}_{a,bulk} \Pi(z, \hat{z}) = -\mathcal{L}_{a,open} \Pi(z, \hat{z}) \quad \xrightarrow{\delta_{\hat{z}} \mathcal{W}=0} \quad \mathcal{L}_{a,bulk} \Pi(z) = f_a(z)\,. \tag{2.19}$$

For the case of the quintic, the inhomogeneous term $f_a(z)$ has been computed by a careful application of the Dwork-Griffiths reduction method for the chain integrals in [9] and it is straightforward to check that this term agrees with the inhomogeneous term on the r.h.s of (2.19), see eq.(3.11) below.

In [28] it has been proposed that the problem of mixed hodge variations on the relative cohomology groups defined in [20, 13] can be reinterpreted in terms of the deformations of a certain non Ricci-flat Kähler blow up $\tilde{Y}$ of the B model geometry. It has been further suggested that it should be possible to obtain a Picard-Fuchs system for the brane geometry by computing in the manifold $\tilde{Y}$ and restricting the complex structure of $\tilde{Y}$ in an appropriate way. At the moment the details appear to be unknown and it would be interesting to relate these ideas to the above results. It would also be interesting to understand a possible connection to the differential equations and superpotentials derived from matrix factorizations in [37, 12, 38].

2.5 Phases of the GLSM and structure of the solutions of (2.18)

In the previous definitions we have used a specific choice of basis $\{l_i^a\}$ to define the local coordinates (2.16) and the differential operators (2.18). Different choices of coordinates correspond to different phases of the GLSM [15]. The extreme cases are on the one hand a large volume phase in all the Kähler parameters, where the GLSM describes a smooth classical geometry and on the other hand a pure Landau-Ginzburg phase. In between there are mixed phases, where only some of the moduli are at large volume and other moduli are fixed in a stringy regime of small volume. A nice instanton expansion can be expected a priori only for moduli at large volume.

Representing the GLSM by the toric polyhedron $\Delta_\flat$, the different phases of the GLSM may be studied by considering different triangulations of the polyhedron [16, 17]. Without going into the technical details of this procedure, let us outline the relevance of this phase structure in the present context. A given B brane configuration corresponds to a critical point of the superpotential which lies in a certain local patch of the parameter space. To study the critical points in a given patch and to give a nice local expansion of the superpotential it is necessary to work in the appropriate local coordinates. The different triangulations of $\Delta_\flat$ define different regimes in the parameter space, where the relative periods Π_Σ have a certain characteristic behavior depending on whether the brane moduli are at large or at small volume. To find an interesting instanton expansion we look for triangulations that correspond to patches where at least some of the moduli are at large volume.

From the interpretation of the system $\{\mathcal{L}_a\}$ of differential operators as the Picard-Fuchs system for the relative periods on Z^* we expect the solutions of the equations (2.17) to have the following structure:

a) There are $2M+2$ solutions $\Pi(z)$ that represent the periods of Z^* up to linear combination and depend only on the complex structure moduli z_a, $a=1,\dots,h^{1,1}(Z)$ of Z^*.

b) There are $2N$ further solutions $\hat{\Pi}(z,\hat{z})$ that do depend on all deformations and define the mirror map for the open string deformations and the superpotential (more precisely: brane tensions).

c) [15] For a maximal triangulation corresponding to a large complex structure point centered at $z_a=0\ \forall a$, there will be a series solution $\omega_0(z_a)=1+\mathcal{O}(z_a)$ and $M+N$ solutions $\omega_c(z_a)$ with a single log behavior that define the open/closed mirror maps as (c is fixed in the following equation)

$$t_c(z_a)=\frac{\omega_c(z_a)}{\omega_0(z_a)}=\frac{1}{2\pi i}\ln(z_c)+S_c(z_a),$$

where $S_c(z_a)$ is a series in the coordinates z_a.

It follows from a) that the mirror map $t^{(cl)}(z)$ in the closed string sector does not involve the open string deformations, similarly as has been observed in [4, 39, 20] in the non-compact case.[16] However the open string mirror map $t^{(op)}(z,\hat{z})$ depends on both types of moduli. For explicit computations of the mirror maps at various points in the moduli we refer to the examples.

The special solution $\Pi=\mathcal{W}_{\rm open}(z,\hat{z})$ has the further property that its instanton expansion near a large volume/large complex structure point encodes the Ooguri-Vafa invariants

[15] The following holds for appropriate choices of normalization and the sign in (2.16) that have been made in (2.18), explaining the special appearance of the entry $i=0$ corresponding to the fiber of the anti-canonical bundle.

[16] This statement holds at zero string coupling.

$\Delta(Z)$	$\nu_0 =$	(0, 0, 0, 0, 0)
	$\nu_1 =$	(- 1, 0, 0, 0, 0)
	$\nu_2 =$	(0, -1, 0, 0, 0)
	$\nu_3 =$	(0, 0, -1, 0, 0)
	$\nu_4 =$	(0, 0, 0, -1, 0)
	$\nu_5 =$	(1, 1, 1, 1, 0)
$\Delta_\flat(Z, L) = \Delta\cup$	$\rho_1 =$	(-1, 0, 0, 0, -1)
	$\rho_2 =$	(0, 0, 0, 0, -1)

Table 1. Points of the enhanced polyhedron $\Delta_\flat$ for the geometry (3.1) on $\mathbf{X}_5^{(1,1,1,1,1)}$.

of the brane geometry:

$$\mathcal{W}_{\text{inst}}(q_a) = \sum_\beta G_\beta q^\beta = \sum_\beta \sum_{k=1}^\infty N_\beta \frac{q^{k.\beta}}{k^2}. \tag{2.20}$$

Here β is the non-trivial homology class of a disc, $\beta \in H^2(Z, L)$, q^β a weight factor related to its appropriately defined Kähler volume, G_β the fractional Gromov-Witten type coefficients in the instanton expansion and N_β the integral Ooguri-Vafa invariants [2].

Below we study some illustrative examples and find agreement with the above expectations.

3 Applications

In the following we apply the above method to study some examples including involution branes with obstructed deformations as well as a class of branes with classically unobstructed moduli.

3.1 Branes on the quintic $\mathbf{X}_5^{(1,1,1,1,1)}$

3.1.1 Brane geometry

We first study a family of toric branes on the quintic that includes branes that have been studied before in [8, 9, 13] by different means. We recover these results for special choice of boundary conditions and study connected configurations. As in section 2. we consider a one parameter family of A branes defined by the two charge vectors

$$(l^1) = (-5, 1, 1, 1, 1, 1), \qquad (\hat{l}^2) = (1, -1, 0, 0, 0, 0)\ . \tag{3.1}$$

As discussed in section 2.3 we may associate with this brane geometry a five-dimensional toric polyhedron $\Delta_\flat(Z, L)$ that contains the points of the polyhedron $\Delta(Z)$ of the quintic as a subset on the hypersurface $y_5 = 0$; these points are given in table 1.

Choosing a maximal triangulation of $\Delta_\flat(Z, L)$ determines the following basis of generators for the relations (2.15)[17]

$$l^1 = (-4, 0, 1, 1, 1, 1; 1, -1), \qquad l^2 = (-1, 1, 0, 0, 0, 0; -1, 1)\,, \tag{3.2}$$

[17] The following computations have been performed using parts of existing computer codes [40].

where the last two entries correspond to the extra points. In the local variables[18]

$$z_1 = -\frac{a_2 a_3 a_4 a_5 a_6}{a_0^4 a_7}, \qquad z_2 = -\frac{a_1 a_7}{a_0 a_6}, \tag{3.3}$$

the hypersurface equations for the B brane geometry (Z^*, E) read

$$\begin{aligned} p(Z^*): &\quad x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - x_1 x_2 x_3 x_4 x_5 \, z^{-\frac{1}{5}} = 0, \\ \mathcal{B}(E): &\quad x_1^5 + x_1 x_2 x_3 x_4 x_5 \; z_2 z^{-\frac{1}{5}} = 0. \end{aligned} \tag{3.4}$$

Here $z = -z_1 z_2$ denotes the complex structure modulus of the CY geometry Z^*.

From eq. (3.2) one can immediately proceed and solve the toric Picard-Fuchs system (2.18) to derive the mirror maps and the superpotentials and we will do so momentarily. However it is instructive to take a closer look at the geometry of the problem of mixed Hodge variations on the relative cohomology groups (2.11), which has the following intriguing structure. Rewriting the superpotential $p(Z^*)$ in the original variables y_i of the toric ambient space and restricting to the hypersurface $\mathcal{B}(E)$: $y_1 = y_0$ in these variables (cpw. (2.4)) defines the following boundary superpotential $W_{\mathcal{H}} = p(Z^*)|_{y_1=y_0}$ for the relative cohomology problem on $\mathcal{H} = \mathcal{B}(E)$:

$$W_{\mathcal{H}} = (a_0 + a_1) y_0 + a_2 y_2 + a_3 y_3 + a_4 y_4 + a_5 y_5 \ .$$

The boundary superpotential $W_{\mathcal{H}}$ describes a K3 surface defined as a quartic polynomial in $\mathbf{P}^3$ after the transformation of variables $y_i = x_i^4$, $i = 1, \ldots, 4$:

$$W_{\mathcal{H}} = x_1^4 + x_2^4 + x_3^4 + x_4^4 + z_{\mathcal{H}}^{-1/4} \; x_1 x_2 x_3 x_4 \ . \tag{3.5}$$

Thus the part of the Hodge variation associated with the lower row in (2.11), which can be properly defined as a subspace through the weight filtration [20, 13], is the usual Hodge variation associated with the complex structure of the family of K3 manifolds defined by $W_{\mathcal{H}}$. The complex structure determined by the (2,0) form ω on the K3 is parametrized by the modulus

$$z_{\mathcal{H}} = \frac{z_1}{(1+z_2)^4} \xrightarrow{a_6/a_7=-1} \frac{a_2 a_3 a_4 a_5}{(a_0+a_1)^4} \,,$$

which is a special combination of the closed and open string moduli. Since the dependence of the Hodge variation on the brane modulus z_2 localizes on $\mathcal{H}$, the open string mirror map and the brane tension will be directly related to periods on the K3 surface (3.5)! This observation is very useful in studying details of the critical points and generalizes to other brane geometries [22].

The differential operators (2.18) in the local variables z_1, z_2 defined by (3.2) read

$$\begin{aligned} \mathcal{L}_1 &= \left(\theta_1^4 - z_1 \prod_{i=1}^{4} (4\theta_1 + \theta_2 + i)\right)(\theta_1 - \theta_2) \,, \\ \mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1 + \theta_2 + 1))(\theta_1 - \theta_2) \ . \end{aligned} \tag{3.6}$$

[18] We equipped z_1 with an additional minus sign compared to (2.16) for later convenience.

JHEP09(2009)126

The above operators $\mathcal{L}_1$ and $\mathcal{L}_2$ reveal the relation of the variation of mixed Hodge structure to the family of K3 manifolds defined in (3.5). Indeed the combination $(\theta_1 - \theta_2)$ is the direction of the open string parameter that localizes on $\mathcal{H}$. The split

$$\mathcal{L}_a = \tilde{\mathcal{L}}_a(\theta_1 - \theta_2)\,,$$

shows that the solutions π_σ of the equations $\tilde{\mathcal{L}}_a \pi_\sigma = 0$ are just the K3 periods. The operator $\tilde{\mathcal{L}}_2$ imposes that the periods depend non-trivially only on the variable $z_{\mathcal{H}}$[19]

$$\tilde{\mathcal{L}}_2\left((z_2+1)^{-1} f(z_{\mathcal{H}})\right) = 0\,,$$

whereas the operator $\tilde{\mathcal{L}}_1$ reduces to the Picard-Fuchs operator of the K3 surface in the new variable $z_{\mathcal{H}}$. It follows that the solutions of the K3 system are the first variations of the relative periods w.r.t. the open string deformation and a critical point $\hat{\delta} W = 0$ corresponds to a particular solution π of the K3 system that vanishes at that point. The solution that describes the involution brane is determined by requiring the right transformation property under the discrete symmetry of the moduli space as in [13].

Further differential operators can be obtained from linear combinations of the basis vectors l^a. e.g. the linear combination $l = l^1 + l^2$ defines the differential operator

$$\mathcal{L}_1' = \theta_2 \theta_1^4 + z_1 z_2 \prod_{i=1}^{5} (4\theta_1 + \theta_2 + i)\,,$$

which also annihilates the relative periods.[20] The solutions of the complete system of differential operators have the expected structure described in section 2.5. The mirror maps can be computed to be

$$\begin{aligned} -\,z_1(t_1,t_2) &= q_1 + (24\,q_1^2 - q_1\,q_2) + (-396\,q_1^3 - 640\,q_1^2\,q_2) + \cdots\,, \\ z_2(t_1,t_2) &= q_2 + (-24\,q_1\,q_2 + q_2^2) + (972\,q_1^2\,q_2 - 178\,q_1\,q_2^2 + q_2^3) + \cdots\,, \end{aligned} \qquad (3.7)$$

with $q_a = \exp(2\pi i\, t_a)$. The deformation parameters t_1 and t_2 are the flat coordinates near the large complex structure point $z_1 = z_2 = 0$ associated with *open string* deformations [20]. Their physical interpretation is the quantum volume of two homologically distinct discs as measured by the tension of D4 domain walls on the A model side [4, 39]. The other solutions of the differential operators (2.18) describe the brane tensions (2.13) of the domain walls in the family. We proceed with a study of various critical points of the superpotential.

3.1.2 Near the involution brane

To study brane configurations mirror to the involution brane we consider a critical point of the type (2.9), that is a D5 brane locus

$$x_2^5 + x_3^5 = 0, \qquad x_4^5 + x_5^5 = 0, \qquad x_1^5 - x_1 x_2 x_3 x_4 x_5\, z^{-\frac{1}{5}} = 0\,.$$

[19]The z_2 dependent prefactor arises from the normalization of the holomorphic form.

[20]One can further factorize the above operators to a degree four differential operator which together with $\mathcal{L}_2$ represents a complete Picard-Fuchs system.

Comparing with (3.4) we search for a superpotential with critical locus near $z_2 = -1$ and arbitrary z_1. Let us first look at the large volume phase $z_1 \sim 0$ of the mirror A brane, where one expects an instanton expansion with integral coefficients. The local variables (3.3) are centered at $z_1 = z_2 = 0$, not $z_2 = -1$, however. To get a nice expansion of the superpotential near the locus $z_2 + 1 = 0$ we change variables to

$$u = z_1^{-1/4}(1 + z_2), \qquad v = z_1^{1/4} .$$

Examining the z_2-dependent solution of the GKZ system in these variables, we find the superpotential

$$c\,\mathcal{W}(u,v) = \frac{u^2}{8} + 15v^2 + \frac{5u^3v}{48} - \frac{15uv^3}{2} + \frac{u^6}{46080} + \frac{35v^2u^4}{384} - \frac{15v^4u^2}{8} + \frac{25025v^6}{3} + \dots \quad (3.8)$$

which has the expected critical locus $\hat{\delta}\mathcal{W} = 0$ at $u = 0$ for all values of v. Here c is a constant that can not be fixed from the consideration of the differential equations (2.18) alone.[21] At the critical locus $u = 0$ the above expression yields the critical value $\mathcal{W}_{\rm crit}(z) = \mathcal{W}(u = 0, v = z^{1/4})$

$$\mathcal{W}_{\rm crit}(z) = 15\sqrt{z} + \frac{25025}{3}\, z^{3/2} + \frac{52055003}{5}\, z^{5/2} + \dots . \qquad (3.9)$$

Here the constant has been fixed to $c = 1$ by comparing (3.9) with the result of [8] for $\mathcal{W}_{\rm crit}(z)$.

As alluded to in section 2.5, the differential operators (3.6) have the special property that the periods of Z^* are amongst their solutions. One may check that the open string mirror maps (3.7) conspire such that the mirror map for the remaining modulus $z = -z_1z_2$ at the critical point coincides with the closed string mirror map for the quintic. Using the multi-cover prescription of [2, 8] and expressing (3.9) in terms of the exponentials $q(z) = \exp(2\pi i\, t(z)) = z + \mathcal{O}(z^2)$ one obtains the integral instanton expansion of the A model

$$\begin{aligned}\frac{\mathcal{W}_{\rm crit}(z(q))}{\omega_0(q)} &= 15\sqrt{q} + \frac{2300}{3}\, q^{3/2} + \frac{2720628}{5}\, q^{5/2} + \dots ,\\ &= \sum_{k\ odd}\left(\frac{15}{k^2}q^{k/2} + \frac{765}{k^2}q^{3k/2} + \frac{544125}{k^2}q^{5k/2} + \dots\right).\end{aligned}$$

To make contact with the inhomogeneous Picard-Fuchs equation of [9], we rewrite the differential operators above in terms of the bulk modulus z and the open string deformation z_2 and split off the z_2 dependent terms as in (2.19). In particular the operator $\mathcal{L}'_1$ leads to a non-trivial equation of the form $\theta\mathcal{L}_{\rm bulk}\Pi = -\mathcal{L}_{\rm open}\Pi$, where

$$\begin{aligned}\mathcal{L}_{\rm bulk} &= \theta^4 - 5z\prod_{i=1}^{4}(5\theta + i)\,, \qquad \mathcal{L}_{\rm open} = \mathcal{L}'_1 - \theta\mathcal{L}_{\rm bulk}\,,\\ \mathcal{L}'_1 &= (\theta + \theta_2)\theta^4 - z\prod_{i=1}^{5}(5\theta + \theta_2 + i)\,,\end{aligned} \qquad (3.10)$$

[21] The precise linear combination of the solutions of the Picard-Fuchs system that corresponds to a given geometric cycle can be determined by an intersection argument and possibly analytic continuation, similarly as in the closed string case [3]. Such an argument has been made in the present example already in [8], from which we will borrow the correct value for c.

and $\theta = \theta_z$. Setting $\Pi = \mathcal{W}(u,v)$ and restricting to the critical locus $z_2 = -1$ one obtains

$$\mathcal{L}_{\rm bulk}\mathcal{W}_{\rm crit} = \frac{15}{16}\sqrt{z}\,. \tag{3.11}$$

This identifies the inhomogeneous Picard-Fuchs equation of [8, 9] as the restriction of (2.18) to the critical locus.

While the result (3.9) had been previously obtained in [8], the above derivation gives some extra information. Since the definition of the toric branes holds off the involution locus, the superpotential $\mathcal{W}(u,v)$ describes more generally any member of the family of toric A branes defined by (2.2), not just the involution brane. It describes also the deformation of the large volume superpotential away from $z_2 = -1$. It is also possible to describe more general configurations with several deformations [22]. It should also be noted that the use of the closed string mirror map in [8] was strictly speaking an assumption, as the closed string mirror map measures the quantum volume of fundamental sphere instantons, not the quantum tension of D4 domain walls wrapping discs, which is the appropriate coordinate for the integral expansion of [2]. It is neither obvious nor true in general that this D4 tension agrees with half the sphere volume of the fundamental string, in particular off the involution locus. In the present case it is not hard to justify this choice and to check it from the computation of the mirror map, but more generally there will be corrections to the D4 quantum volume that are not determined by the closed string mirror map, see eq. (3.7) and the examples below.

Small volume in the A model: $1/z_1 \sim 0$. Another interesting point in the moduli space is the Landau-Ginzburg point of the B model. This case has been studied previously in [13], so we will be very brief. The only non-trivial thing left to check is that the system of differential equations obtained in [13] from Dwork-Griffiths reduction is equivalent to the toric GKZ system (2.18) transformed to the local variables near the LG point. Choosing local variables

$$x_1 = \frac{a_0}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{1/4}, \qquad x_2 = \frac{a_1}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{5/4},$$

one obtains by a transformation of variables the differential operators

$$\begin{aligned}
\mathcal{L}_1 &= \left(x_1^4(\theta_1+\theta_2)^4 - 4^4\prod_{i=1}^{4}(\theta_1 - i)\right)(\theta_1 + 5\theta_2)\,,\\
\mathcal{L}_2 &= \left(x_2(\theta_1-1) - x_1\theta_2\right)(\theta_1+5\theta_2)\,,\\
\mathcal{L}_1' &= x_1^5(\theta_1+\theta_2)^4\theta_2 - 4^4 x_2\prod_{i=1}^{5}(\theta_1 - i)\,,
\end{aligned} \tag{3.12}$$

where θ_i denotes the logarithmic derivatives θ_{x_i}. The above operators agree with eqs. (5.14)–(5.16) of [13] up to a change of variables. The superpotential is

$$\mathcal{W} = -\frac{x_1^2}{2} - \frac{x_2x_1}{6} - \frac{x_1^6}{11520} - \frac{x_2x_1^5}{3840} - \frac{x_2^2x_1^4}{2688} - \frac{x_2^3x_1^3}{3456} - \frac{x_2^4x_1^2}{8448} - \frac{x_2^5x_1}{49920} + \ldots\,,$$

which has its critical locus at $x_2 = -x_1$, which corresponds to $u = 0$ in these coordinates. In terms of the closed string variable $x = -x_1 x_2^{-1/5}$ at the Landau Ginzburg point, the expansion at the critical locus reads

$$\mathcal{W}_{\mathrm{crit}} = -\frac{x^{5/2}}{3} - \frac{x^{15/2}}{135135} - \frac{x^{25/2}}{1301375075} + \dots ,$$

which satisfies a similar equation as (3.11)

$$\mathcal{L}_{\mathrm{bulk}} \mathcal{W}_{\mathrm{crit}} = \frac{15}{16} x^{5/2} ,$$

where $\mathcal{L}_{\mathrm{bulk}} = 5^{-4} x^5 \theta_x^4 - 5 \prod_{i=1}^{4} (\theta_x - i)$.

3.2 Branes on $\mathbf{X}_{18}^{(1,1,1,6,9)}$

As a second example we study branes on the two moduli CY $Z = \mathbf{X}_{18}^{(1,1,1,6,9)}$. Z is an elliptic fibration over $\mathbf{P}^2$ with the elliptic fiber and the base parametrized by the coordinates x_1, x_2, x_3 and x_4, x_5, x_6 in (2.1), respectively. In the decompactification limit of large fiber, the compact CY approximates the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$ with coordinates x_3, x_4, x_5, x_6. This limit is interesting, as it makes contact to the previous studies of branes on $\mathcal{O}(-3)_{\mathbf{P}^2}$ in [39, 19].

3.2.1 Brane geometry

We consider a family of A branes parametrized by the relations

$$|x_4|^2 - |x_3|^2 = c^1, \qquad \hat{l} = (0, 0, 0, -1, 1, 0, 0) \ . \tag{3.13}$$

This defines a family of D7-branes in the mirror parametrized by one complex modulus. To make contact with the non-compact branes we may add a second constraint $|x_5|^2 - |x_3|^2 = 0$ that selects a particular solution of the Picard-Fuchs system.[22] The brane geometry on the B model side is defined by the two equations

$$\begin{aligned} p(Z^*) = \sum a_i y_i \ &= a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^2 + a_2 x_2^3 + a_3 (x_3 x_4 x_5)^6 + a_4 x_3^{18} + a_5 x_4^{18} + a_6 x_5^{18}, \\ \mathcal{B}(E) : \qquad y_3 \ &= y_4 \qquad \text{or} \qquad (x_3 x_4 x_5)^6 = x_3^{18} \ . \end{aligned} \tag{3.14}$$

As in the previous case one observes that the complex deformations of the brane geometry are related to the periods of a K3 surface defined by

$$W_{\mathcal{H}} = a_0 x_1' x_2' x_3' x_4' + a_1 {x_1'}^2 + a_2 {x_2'}^3 + (a_3 + a_4)(x_3' x_4')^6 + a_5 {x_3'}^{12} + a_6 {x_4'}^{12} \ .$$

The GLSM for the above brane geometry corresponds to the enhanced polyhedron given in table 2.

Choosing a triangulation of $\Delta_\flat$ that represents a large complex structure phase yields the following basis of the linear relations (2.15) between the points of $\Delta_\flat$:

$$l^1 \ = (-6, 3, 2, 1, 0, 0, 0, 0, 0),$$

[22] Since the constant in this equation must be zero to get a non-zero superpotential [4], there is no new modulus.

$\Delta(Z)$	$\nu_0 =$	(0, 0, 0, 0, 0)
	$\nu_1 =$	(0, 0, 0, -1, 0)
	$\nu_2 =$	(0, 0, -1, 0, 0)
	$\nu_3 =$	(0, 0, 2, 3, 0)
	$\nu_4 =$	(-1, 0, 2, 3, 0)
	$\nu_5 =$	(0, -1, 2, 3, 0)
	$\nu_6 =$	(1, 1, 2, 3, 0)
$\Delta_\flat(Z,E) = \Delta\cup$	$\rho_1 =$	(0, 0, 2, 3, -1)
	$\rho_2 =$	(-1, 0, 2, 3, -1)

Table 2. Points of the enhanced polyhedron $\Delta_\flat$ for the geometry (3.13) on $\mathbf{X}_{18}$.

$$\begin{aligned} l^2 &= (0,0,0,-2,0,1,1,-1,1), \\ l^3 &= (0,0,0,-1,1,0,0,1,-1). \end{aligned} \tag{3.15}$$

The last two charge vectors define a GLSM for the "inner phase" of the brane in the non-compact CY described in [19]. The differential operators (2.18) for the relative periods are given by

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - 2\theta_2 - \theta_3) - 12z_1(6\theta_1+5)(6\theta_1+1), \\ \mathcal{L}_2 &= \theta_2^2(\theta_2-\theta_3) + z_2(\theta_1-2\theta_2-\theta_3)(\theta_1-2\theta_2-1-\theta_3)(\theta_2-\theta_3), \\ \mathcal{L}_3 &= -\theta_3(\theta_2-\theta_3) - z_3(\theta_1-2\theta_2-\theta_3)(\theta_2-\theta_3). \end{aligned} \tag{3.16}$$

3.2.2 Large volume brane

The elliptic fiber compactifies the non-compact fiber direction x_3 of the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$. In the limit of large elliptic fiber we therefore expect to find a deformation of the brane studied in [39, 19]. Large volume corresponds to $z_a = 0$ in the coordinates defined by eqs. (3.15), (2.16).

The mirror maps and the superpotential can be computed from (2.18). Expressing the superpotential in the flat coordinates t_a defines the Ooguri-Vafa invariants N_β in (2.20). The homology class β can be labelled by three integers (k,l,m) that determine the Kähler volume $kt_1 + lt_2 + mt_3$ of a curve in this class. Here t_1 is the volume of the elliptic fiber and t_2, t_3 are the (D4-)volumes of two homologically distinct discs in the brane geometry. The Kähler class of the section, which measures the volume of the fundamental sphere in $\mathbf{P}^2$, is $t_2 + t_3$.

For the discs that do not wrap the elliptic fiber we obtain for $\beta = (0,l,m)$ the invariants given in table 3.

The above result agrees with the results of [39, 19] for the disc invariants in the "inner phase" of the non-compact CY $\mathcal{O}(-3)_{\mathbf{P}^2}$. This result can be explained heuristically as follows. The holomorphic discs ending on the non-compact A brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$ lie within the zero section of $\mathcal{O}(-3)_{\mathbf{P}^2}$. Similarly discs with $k=0$ in $\mathbf{X}_{18}$ are holomorphic curves that must map to the section $x_3 = 0$ of the elliptic fibration. The moduli space of maps into the sections of the non-compact and compact manifolds, respectively, does not

l \ m	0	1	2	3	4	5	6
0	*	1	0	0	0	0	0
1	1	*	-1	-1	-1	-1	-1
2	-1	-2	*	5	7	9	12
3	1	4	12	*	-40	-61	-93
4	-2	-10	-32	-104	*	399	648
5	5	28	102	326	1085	*	-4524
6	-13	-84	-344	-1160	-3708	-12660	*
7	35	264	1200	4360	14274	45722	159208
8	-100	-858	-4304	-16854	-57760	-185988	-598088
9	300	2860	15730	66222	239404	793502	2530946
10	-925	-9724	-58208	-262834	-1004386	-3460940	-11231776

Table 3. Invariants $N_{0,l,m}$ for the geometry (3.15).

l \ m	0	1	2	3	4	5
0	*	252	0	0	0	0
1	-240	*	300	300	300	300
2	240	780	*	-2280	-3180	-4380
3	-480	-2040	-6600	*	24900	39120
4	1200	6300	22080	74400	*	-315480
5	-3360	-21000	-82200	-276360	-957600	*
6	10080	73080	319200	1134000	3765000	13300560
7	-31680	-261360	-1265040	-4818240	-16380840	-54173880

Table 4. Invariants $N_{1,l,m}$ for the geometry (3.15).

l \ m	0	1	2	3	4
0	*	5130	-18504	0	0
1	-141444	*	-73170	-62910	-62910
2	-28200	-108180	*	544140	778560
3	85320	403560	1557000	*	-7639920
4	-285360	-1647540	-6485460	-24088680	*
5	1000440	6815160	29214540	106001100	392435460
6	-3606000	-28271880	-133294440	-505417320	-1773714840

Table 5. Invariants $N_{2,l,m}$ for the geometry (3.15).

the compactification in the fiber, explaining the agreement. The agreement of the two computations can be viewed as a statement of local mirror symmetry in the open string setup. For world-sheets that wrap the fiber we obtain the invariants given in tables 4 and 5.

It would be interesting to confirm some of these numbers by an independent computation.

3.2.3 Deformation of the non-compact involution brane

In [10] an involution brane in the local model $\mathcal{O}(-3)_{\mathbf{P}^2}$ has been studied. Similarly as in the previous case one expects to find a deformation of this brane by embedding it in the compact manifold and taking the limit of large elliptic fiber, $z_1 = 0$. In order to recover the involution brane of the local geometry we study the critical points near $z_3 = 1$ in the local coordinates

$$\tilde{z}_1 = z_1(-z_2)^{1/2}, \quad u = (-z_2)^{-1/2}(1-z_3), \quad v = (-z_2)^{1/2}.$$

After transforming the Picard-Fuchs system to these variables, the solution corresponding to the superpotential has the following expansion

$$c\mathcal{W} = -v - \frac{35v^3}{9} + \frac{1}{2}uv^2 + \frac{200}{3}\tilde{z}_1 v^2 - \frac{u^2 v}{8} - 12320\tilde{z}_1^2 v - 60u\tilde{z}_1 v + \dots, \tag{3.17}$$

where c is a constant that will be fixed again by comparing the critical value with the results of [10]. In the decompactification limit $\tilde{z}_1 = 0$, the critical point of the superpotential is at $u = 0$, where we obtain the following expansion

$$c\mathcal{W}|_{\text{crit}} = -\sqrt{z_2} - \frac{35}{9}z_2^{3/2} - \frac{1001}{25}z_2^{5/2} + \dots, \tag{3.18}$$

The restricted superpotential satisfies the differential equation

$$\mathcal{L}_{\text{bulk}}\mathcal{W}|_{\text{crit}} = -\frac{\sqrt{z_2}}{8c},$$

with $\mathcal{L}_{\text{bulk}}$ the Picard-Fuchs operator of the local geometry $\mathcal{O}(-3)_{\mathbf{P}^2}$. The above expressions at the critical point agree with the ones given in [10] for $c = 1$.

As might have been expected, the full superpotential (3.17) shows that the involution brane of the local model is non-trivially deformed in the compact CY manifold for $z_1 \neq 0$. It is not obvious that the modified multi-cover description of [8], which is adapted to real curves and differs from the original proposal of [2], can be generalized to obtain integral invariants for the deformations of the critical point in the z_1 direction. One suspects that an integral expansion in the sense of [8] exists only at critical points with an extra symmetry and for deformations that respect this symmetry. It will be interesting to study this further.

3.3 Branes on $\mathbf{X}_9^{(1,1,1,3,3)}$

As a third example we study branes on the two moduli CY $Z = \mathbf{X}_9^{(1,1,1,3,3)}$. Z is again an elliptic fibration over $\mathbf{P}^2$ and one can consider a similar compactification of the non-compact brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$. The invariants for this geometry are reported in app. B.

Here we consider a different family of D7-branes which we expect to include a brane that exists at the Landau Ginzburg point of the two moduli Calabi-Yau. The mirror A brane is defined by

$$-|x_0|^2 + |x_1|^2 = c^1, \qquad \hat{l} = (-1,1,0,0,0,0,0)\ . \tag{3.19}$$

$\Delta(Z)$	$\nu_0 =$	(0, 0, 0, 0, 0)
	$\nu_1 =$	(0, 0, 0, -1, 0)
	$\nu_2 =$	(0, 0, -1, 0, 0)
	$\nu_3 =$	(0, 0, 1, 1, 0)
	$\nu_4 =$	(-1, 0, 1, 1, 0)
	$\nu_5 =$	(0, -1, 1, 1, 0)
	$\nu_6 =$	(1, 1, 1, 1, 0)
$\Delta_\flat(Z,E) = \Delta\cup$	$\rho_1 =$	(0, 0, 0, 0, -1)
	$\rho_2 =$	(0, 0, 0,- 1, -1)

Table 6. Points of the enhanced polyhedron $\Delta_\flat$ for the geometry (3.19).

The polyhedron for the GLSM is given in table 6.

A suitable basis of relations for the charge vectors is

$$\begin{aligned} l^1 &= (-2,0,1,1,0,0,0,-1,1), \quad l^2 = (0,0,0,-3,1,1,1,0,0), \\ l^3 &= (-1,1,0,0,0,0,0,1,-1)\,, \end{aligned} \tag{3.20}$$

leading to the differential operators

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1-3\theta_2)(\theta_1-\theta_3)+z_1(\theta_1-\theta_3)(2\theta_1+1+\theta_3)(2\theta_1+2+\theta_3), \\ \mathcal{L}_2 &= \theta_2^3-z_2(\theta_1-3\theta_2)(\theta_1-3\theta_2-1)(\theta_1-3\theta_2-2), \\ \mathcal{L}_3 &= -\theta_3(\theta_1-\theta_3)-z_3(\theta_1-\theta_3)(2\theta_1+1+\theta_3)\;. \end{aligned} \tag{3.21}$$

The brane geometry on the B model side is defined by the two equations

$$\begin{aligned} p(Z^*) = \sum a_i y_i &= a_0x_1x_2x_3x_4x_5+a_1x_1^3+a_2x_2^3+a_3(x_3x_4x_5)^3+a_4x_3^9+a_5x_4^9+a_6x_5^9, \\ \mathcal{B}(E): \qquad y_0 &= y_1 \qquad \text{or} \qquad x_1x_2x_3x_4x_5 = x_1^3. \end{aligned} \tag{3.22}$$

As in the previous cases, the deformations of the hypersurface $\mathcal{B}(E)$ are described by the periods on a K3 surface.

We are interested in a brane superpotential with critical point at $z_3=-1$. Choosing the following local coordinates centered around $z_3=-1$

$$u=(-z_1)^{-1/2}z_2^{-1/6}(z_3+1)\,,\quad v=(-z_1)^{1/2}z_2^{1/6}\quad x_2=z_2^{-1/3}\,,$$

we obtain the superpotential

$$c\mathcal{W} = -\frac{1}{2}ux_2+\frac{1}{24}u^3+210v^3+\frac{3}{4}vx_2^2-\frac{3}{8}u^2vx_2+\dots\,. \tag{3.23}$$

This superpotential has a critical point at $u=0$ and $x_2=0$. At the critical locus we have $v=z^{1/6}$, where z denotes the closed string modulus

$$z=-\frac{a_1^3a_2^3a_4a_5a_6}{a_0^9}\,.$$

The expansion of the superpotential at this critical locus reads

$$c\,\mathcal{W}|_{\text{crit}} = 210\sqrt{z}+\frac{53117350}{3}z^{3/2}+\frac{18297568296042}{5}z^{5/2}+\frac{7182631458065952702}{7}z^{7/2}+\dots\,,$$

As in the example of section 3.2.3 it is an interesting question to study the instanton expansion of the above expressions and its possible interpretation in terms of integral BPS invariants. We leave this for the future.

4 Summary and outlook

As proposed above, the open/closed string deformation space of the toric branes defined in [4] can be studied by mirror symmetry and toric geometry in a quite efficient way. The toric definition of the brane geometry in section 2 leads to the canonical Picard-Fuchs system (2.18), whose solutions determine the mirror maps and the superpotential. The phase structure of the associated GLSM determines large volume regimes, where the superpotential has an disc instanton expansion with an interesting mathematical and physical interpretation.

Since the toric branes cover only a subset within the category of D-branes, e.g. matrix factorizations on the B model side, it is natural to ask for the precise relation between these two definitions. It is an interesting question to which extent it is possible to lift the machinery of toric geometry directly to the matrix factorization and to make contact with the works [37, 38]. On the positive side one notices that the class of toric branes is already rather large and not too special, as can be seen from the fact that the above framework covers all cases where explicit results have been obtained so far.

There are some other obvious questions left open by the above discussion, such as the geometric and physical interpretation of some of the objects appearing in the definition of the GLSM and the mirror B geometry, e.g. the appearance of the "enhanced polyhedra" $\Delta_\flat(Z,L)$ and K3 surfaces, which beg for an explanation. A discussion of these issues is beyond the scope of this paper and will be given elsewhere [22], but here we outline some of the answers. As the reader may have noticed, the polyhedra $(\Delta_\flat(Z,L),\Delta_\flat^\star(Z^*,E))$ define Calabi-Yau fourfolds, which are the hallmark of F-theory compactifications with the same supersymmetry.[23] Another conclusive hint towards F-theory comes from the fact that we have effectively studied families of 7-branes on the B model side by intersecting a single equation with the Calabi-Yau hypersurface. In fact the "auxiliary geometry" defined in section 2.3 should be viewed as a physical 7-brane geometry and this interpretation suggests that the results of the GLSM determine also the Kähler metric on the open/closed deformation space.

Acknowledgments

We are indebted to Hans Jockers for discussions and exchange of ideas. We would also like to thank Marco Baumgartl, Ilka Brunner, Thomas Grimm, Albrecht Klemm, Johanna

[23] An M-theory interpretation of the 4-folds for local models has been given in [36]. The third author thanks M. Aganagic and C. Vafa for pointing out a possible F-theory interpretation.

Knapp, Christian Römelsberger and Emanuel Scheidegger for discussions and comments. The work of M.A. and P.M. is supported by the program “Origin and Structure of the Universe” of the German Excellence Initiative. The work of M.H. is supported by the Deutsche Forschungsgemeinschaft.

A One parameter models

In the following we discuss the toric GKZ systems associated to brane families connected to the involution brane in one parameter compact models.[24] At the critical value of the superpotential we recover the results of [11, 12].

A.1 Sextic $\mathbf{X}_6^{(2,1,1,1,1)}$

We consider the charge vectors

$$l^1 = (-4, 0, 1, 1, 1, 1; 2, -2), \qquad l^2 = (-1, 1, 0, 0, 0, 0; -1, 1) \ .$$

A.1.1 Large volume

This region in moduli space is parameterized by local variables

$$z_1 = \frac{a_2 a_3 a_4 a_5 a_6^2}{a_0^4 a_7^2}\,, \qquad z_2 = -\frac{a_1 a_7}{a_0 a_6}\,.$$

We obtain the differential operators

$$\begin{aligned}
\mathcal{L}_1 &= \left(\theta_1^4 - z_1 \prod_{i=1}^{4} (4\theta_1 + \theta_2 + i)\right)(2\theta_1 - \theta_2)\,, \\
\mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1 + \theta_2 + 1))(2\theta_1 - \theta_2)\,, \\
\mathcal{L}_1' &= \theta_1^4 \prod_{i=0}^{1} (\theta_2 - i) - z_1 z_2^2 \prod_{i=1}^{6} (4\theta_1 + \theta_2 + i)\,.
\end{aligned}$$

Switching to coordinates which are centered around the critical point $z_2 = -1$ of the superpotential

$$u = z_1^{-1/4}(z_2 + 1)\,, \quad v = z_1^{1/4}\,,$$

we obtain the superpotential

$$c\mathcal{W}(u,v) = \frac{u^2}{24} + 24v^2 + \frac{u^3 v}{24} - 24uv^3 + \frac{u^6}{138240} + \frac{v^2 u^4}{24} + \frac{143360 v^6}{3} + \dots \qquad \text{(A.1)}$$

At the critical point $u = 0$, we can express v in terms of the closed string modulus $z = z_1 z_2^2$ as

$$v|_{\rm crit} = z^{1/4}\,.$$

We find for the superpotential at the minimum

$$c\mathcal{W}_{\rm crit} = 24\sqrt{z} + \frac{143360}{3} z^{3/2} + \frac{5510529024}{25} z^{5/2} + \frac{334766662483968}{245} z^{7/2} + \dots\,,$$

[24] See [41] for a discussion of closed string mirror symmetry in these models.

JHEP09(2009)126

This expression satisfies the differential equation

$$\mathcal{L}_{\rm bulk}\mathcal{W}_{\rm crit} = \frac{3}{2c}\sqrt{z}\,,$$

where $\mathcal{L}_{\rm bulk} = \theta^4 - 9z\prod_{i=1}^4(6\theta+i)$ denotes the Picard-Fuchs operator of the sextic. The above agrees with the results of [11] for the choice of constant $c = 1$.

A.1.2 Small volume

To study the Landau-Ginzburg phase of the B-model we change to the local coordinates

$$x_1 = \frac{a_0}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{1/2}\,, \quad x_2 = \frac{a_1}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{3/2}\,.$$

The differential operators obtained by a transformation of variables are ($\theta_i = \theta_{x_i}$)

$$\begin{aligned}
\mathcal{L}_1 &= \left(x_1^4(\theta_1+\theta_2)^4 - 4^4\prod_{i=1}^4(\theta_1 - i)\right)(\theta_1+3\theta_2)\,,\\
\mathcal{L}_2 &= (x_2(\theta_1-1) - x_1\theta_2)(\theta_1+3\theta_2)\,,\\
\mathcal{L}_1' &= x_1^6(\theta_1+\theta_2)^4\theta_2(\theta_2-1) - 4^4x_2^2\prod_{i=1}^6(\theta_1-i)\,.
\end{aligned}$$

We obtain the superpotential

$$\mathcal{W} = -\frac{1}{12}x_1^2 - \frac{1}{24}x_2x_1 - \frac{x_1^6}{69120} - \frac{x_2x_1^5}{18432} - \frac{x_2^2x_1^4}{11520} - \frac{x_2^3x_1^3}{13824} - \frac{x_2^4x_1^2}{32256} - \frac{x_2^5x_1}{184320} + \ldots \quad \text{(A.2)}$$

which has its critical value at $x_2 = -x_1$. We can express x_1 in terms of the closed string variable $x = -x_1x_2^{-1/3}$ of the geometry in the Landau-Ginzburg phase as

$$x_1|_{\rm crit} = -x^{3/2}$$

which gives the following critical value for the superpotential

$$\mathcal{W}_{\rm crit} = -\frac{x^3}{24} - \frac{x^9}{3870720} - \frac{x^{15}}{137763225600} - \frac{5x^{21}}{16403566461714432} + \ldots$$

This expression satisfies the equation

$$\mathcal{L}_{\rm bulk}\mathcal{W}_{\rm crit} = \frac{3}{2}x^3\,,$$

with $\mathcal{L}_{\rm bulk} = 6^{-4}x^6\theta^4 - 9(\theta-1)(\theta-2)(\theta-4)(\theta-5)\,.$

A.2 Octic

We consider the charge vectors

$$l^1 = (-4,0,1,1,1,1;4,-4), \qquad l^2 = (-1,1,0,0,0,0;-1,1)\,.$$

A.2.1 Large volume

This region in moduli space is parameterized by local variables

$$z_1 = \frac{a_2 a_3 a_4 a_5 a_6^4}{a_0^4 a_7^4}\,, \quad z_2 = -\frac{a_1 a_7}{a_0 a_6}\,.$$

The differential operators are

$$\begin{aligned}
\mathcal{L}_1 &= \left(\theta_1^4 - z_1 \prod_{i=1}^{4}(4\theta_1+\theta_2+i)\right)(4\theta_1-\theta_2)\,,\\
\mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1+\theta_2+1))(4\theta_1-\theta_2)\,,\\
\mathcal{L}_1' &= \theta_1^4 \prod_{i=0}^{3}(\theta_2 - i) - z_1 z_2^4 \prod_{i=1}^{8}(4\theta_1+\theta_2+i)\,.
\end{aligned}$$

Switching to $u = z_1^{-1/4}(z_2+1)$ and $v = z_1^{1/4}$, we obtain

$$\mathcal{W}(u,v) = \frac{u^2}{16} + 48v^2 + \frac{u^3 v}{12} - 96uv^3 + \frac{u^6}{92160} + \frac{5v^2u^4}{48} + 48v^4u^2 + \frac{1576960v^6}{3} + \ldots \quad \text{(A.3)}$$

At $u = 0$, we can express v in terms of the classical coordinate $z = z_1 z_2^4$ as $v|_{\text{crit}} = -z^{1/4}$. We find for the superpotential at the minimum

$$c\mathcal{W}_{\text{crit}} = 48\sqrt{z} + \frac{1576960}{3} z^{3/2} + \frac{339028738048}{25} z^{5/2} + \frac{23098899711393792}{49} z^{7/2} + \ldots\,,$$

which satisfies the differential equation

$$\mathcal{L}_{\text{bulk}} \mathcal{W}_{\text{crit}} = \frac{3}{c}\sqrt{z}\,,$$

where $\mathcal{L}_{\text{bulk}} = \theta^4 - 16z\prod_{i=1}^{4}(8\theta+2i-1)$ denotes the Picard-Fuchs operator of the octic. Setting $c = 1$ reproduces the disk invariants of [11, 12].

A.2.2 Small volume

We switch to local coordinates

$$x_1 = \frac{-a_0 a_7}{a_6 (a_2 a_3 a_4 a_5)^{1/4}}\,, \quad x_2 = \frac{a_1 a_7^2}{a_6^2 (a_2 a_3 a_4 a_5)^{1/4}}\,.$$

The differential operators are ($\theta_i = \theta_{x_i}$)

$$\begin{aligned}
\mathcal{L}_1 &= \left(x_1^4(\theta_1+\theta_2)^4 - 4^4 x_2^2 \prod_{i=1}^{4}(\theta_1 - i)\right)(\theta_1+2\theta_2)\,,\\
\mathcal{L}_2 &= (x_2(\theta_1-1) - x_1\theta_2)(\theta_1+2\theta_2)\,,\\
\mathcal{L}_1' &= x_1^8(\theta_1+\theta_2)^4 \prod_{i=0}^{3}(\theta_2 - i) - 4^4 x_2^2 \prod_{i=1}^{8}(\theta_1 - i)\,.
\end{aligned}$$

$k=0$						
$l \backslash m$	0	1	2	3	4	5
0	*	3	0	0	0	0
1	3	*	-3	-3	-3	-3
2	-3	-6	*	15	21	27
3	3	12	36	*	-120	-183
4	-6	-30	-96	-312	*	1197
5	15	84	306	978	3255	*
6	-39	-252	-1032	-3480	-11124	-37980
7	105	792	3600	13080	42822	137166

	$k=1$				$k=2$			
$l \backslash m$	0	1	2	3	0	1	2	3
0	*	27	0	0	*	81	-108	0
1	-72	*	90	90	-1269	*	-1539	-1377
2	72	234	*	-684	-684	-2808	*	13554
3	-144	-612	-1980	*	2268	11232	42336	*
4	360	1890	6624	22320	-7848	-46656	-182916	-671922
5	-1008	-6300	-24660	-82908	27972	194832	835758	3020382
6	3024	21924	95760	340200	-102024	-813456	-3844512	-14554242
7	-9504	-78408	-379512	-1445472	377784	3390336	17598600	70975872

Table 7. Invariants $N_{k,l,m}$ for the geometry (B.1).

We obtain the superpotential

$$\mathcal{W} = -\frac{1}{16}x_1^2 - \frac{1}{24}x_2x_1 - \frac{x_1^6}{92160} - \frac{x_2x_1^5}{21504} - \frac{x_2^2x_1^4}{12288} - \frac{x_2^3x_1^3}{13824} - \frac{x_2^4x_1^2}{30720} - \frac{x_2^5x_1}{168960} + \dots \quad \text{(A.4)}$$

At the critical value $x_2 = -x_1$, we have $x_1|_{\text{crit}} = -x^2$, where $x = -x_1x_2^{-1/2}$. This gives the following expansion for the superpotential

$$\mathcal{W}_{\text{crit}} = -\frac{x^4}{48} - \frac{x^{12}}{42577920} - \frac{x^{20}}{8475718451200} - \frac{x^{28}}{1131846085858295808} + \dots$$

which satisfies the equation

$$\mathcal{L}_{\text{bulk}}\mathcal{W}_{\text{crit}} = 3x^4\,,$$

with

$$\mathcal{L}_{\text{bulk}} = 8^{-4}x^8\theta^4 - 16(\theta-1)(\theta-3)(\theta-5)(\theta-7)\,.$$

These results are in agreement with [13], where this phase of the moduli space has been previously studied.

m \ k	0	1	2	3	4	5
	$l=0$					
0	*	54	0	0	0	0
1	-36	*	54	-18	0	0
2	18	-54	*	36	0	0
3	0	0	-54	*	54	0
4	0	0	0	-36	*	54
5	0	0	0	18	-54	*
6	0	0	0	0	0	-54
7	0	0	0	0	0	0

	$l=1$					$l=2$			
m \ k	0	1	2	3	4	0	1	2	3
0	*	0	0	0	0	*	0	0	0
1	72	*	-108	36	0	-180	*	270	-90
2	-36	-1728	*	2772	-1026	108	7020	*	-11160
3	-1224	17280	-80460	*	243756	-108	-5832	-97686	*
4	5508	-64800	340092	-1075140	*	-10944	133488	-588276	2643372

Table 8. Invariants $N_{k,l,m}$ for the geometry (3.20).

B Invariants for $\mathbf{X}_9^{1,1,1,3,3}$

The compactification of the local brane in $\mathcal{O}(-3)_{\mathbf{P}^2}$ is described by the charge vectors

$$l^1 = (-3,1,1,1,0,0,0,0,0), l^2 = (0,0,0,-2,0,1,1,-1,1), l^3 = (0,0,0,-1,1,0,0,1,-1) \ . \tag{B.1}$$

Some invariants for this geometry are given in table 7.

The invariants for $k=0$ are three times the invariants in table 3, where the overall factor comes from the three global sections of the elliptic fibration $\mathbf{X}_9$. It appears that the invariants for $k=1,\ l\neq 0$ are generally $3/10$ times the invariants in table 4.

Some invariants for the geometry (3.20) in the large volume phase are given in table 8.

It would be interesting to check some of these predictions by an independent computation.

References

[1] M. Kontsevich, *Homological algebra of mirror symmetry*, *Proc. Internat. Congress Math.* **1** (1995) 120 [alg-geom/9411018].

[2] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys.* **B 577** (2000) 419 [hep-th/9912123] [SPIRES].

[3] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B 359** (1991) 21 [SPIRES].

[4] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, hep-th/0012041 [SPIRES].

[5] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637 [hep-th/9207094] [SPIRES].

[6] E. Witten, *New issues in manifolds of* SU(3) *holonomy*, *Nucl. Phys.* **B 268** (1986) 79 [SPIRES].

[7] M. Mariño, *Chern-Simons theory, matrix models and topological strings*, International Series of Monographs on Physics, Oxford University Press, Oxford U.K. (2005);
W. Lerche, *Special geometry and mirror symmetry for open string backgrounds with* $N = 1$ *supersymmetry*, hep-th/0312326 [SPIRES].

[8] J. Walcher, *Opening mirror symmetry on the quintic*, *Commun. Math. Phys.* **276** (2007) 671 [hep-th/0605162] [SPIRES].

[9] D.R. Morrison and J. Walcher, *D-branes and normal functions*, arXiv:0709.4028 [SPIRES].

[10] J. Walcher, *Evidence for tadpole cancellation in the topological string*, arXiv:0712.2775 [SPIRES].

[11] D. Krefl and J. Walcher, *Real mirror symmetry for one-parameter hypersurfaces*, *JHEP* **09** (2008) 031 [arXiv:0805.0792] [SPIRES].

[12] J. Knapp and E. Scheidegger, *Towards open string mirror symmetry for one-parameter Calabi-Yau hypersurfaces*, arXiv:0805.1013 [SPIRES].

[13] H. Jockers and M. Soroush, *Effective superpotentials for compact D5-brane Calabi-Yau geometries*, *Commun. Math. Phys.* **290** (2009) 249 [arXiv:0808.0761] [SPIRES].

[14] R. Pandharipande, J. Solomon and J. Walcher, *Disk enumeration on the quintic* 3*-fold*, *J. Amer. Math. Soc.* **21** (2008) 1169 [math.SG/0610901].

[15] E. Witten, *Phases of* $N = 2$ *theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042] [SPIRES].

[16] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*, *Nucl. Phys.* **B 416** (1994) 414 [hep-th/9309097] [SPIRES].

[17] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, *J. Alg. Geom.* **3** (1994) 493 [SPIRES].

[18] D.A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**, American Mathematical Society, Providence U.S.A. (2000) [SPIRES];
K. Hori et. al, *Mirror symmetry*, Clay Mathematics Monographs **1**, American Mathematical Society, Providence U.S.A. (2003) [SPIRES].

[19] W. Lerche and P. Mayr, *On* $N = 1$ *mirror symmetry for open type-II strings*, hep-th/0111113 [SPIRES].

[20] W. Lerche, P. Mayr and N. Warner, $N = 1$ *special geometry, mixed Hodge variations and toric geometry*, hep-th/0208039 [SPIRES]; *Holomorphic* $N = 1$ *special geometry of open-closed type-II strings*, hep-th/0207259 [SPIRES].

[21] R. Harvey and B. Lawson, *Calibrated geometries*, *Acta Math.* **148** (1982) 47 [SPIRES].

[22] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush, *Hints for off-shell mirror symmetry in type II/F-theory compactifications*, arXiv:0909.1842 [SPIRES].

JHEP09(2009)126

[23] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222 [SPIRES].

[24] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, hep-th/9804131 [SPIRES].

[25] H. Jockers and W. Lerche, *Matrix factorizations, D-branes and their deformations*, *Nucl. Phys. (Proc. Suppl.)* **171** (2007) 196 [arXiv:0708.0157] [SPIRES];
J. Knapp, *D-branes in topological string theory*, arXiv:0709.2045 [SPIRES].

[26] M. Herbst, K. Hori and D. Page, *Phases of $N = 2$ theories in $1 + 1$ dimensions with boundary*, arXiv:0803.2045 [SPIRES].

[27] B. Forbes, *Open string mirror maps from Picard-Fuchs equations on relative cohomology*, hep-th/0307167 [SPIRES].

[28] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *The D5-brane effective action and superpotential in $N = 1$ compactifications*, *Nucl. Phys.* **B 816** (2009) 139 [arXiv:0811.2996] [SPIRES].

[29] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics **76**, Cambridge University Press, Cambridge U.K. (2002); *Hodge theory and complex algebraic geometry. II*, Cambridge Studies in Advanced Mathematics **77**, Cambridge University Press, Cambridge U.K. (2003);
C.A.M. Peters and J.H.M. Steenbrink, *Mixed Hodge structures*, A Series of Modern Surveys in Mathematics **52**, Springer, U.S.A. (2008).

[30] P. Deligne, *Theorié de Hodge I* (in French), in *Actes de Congrès international de Mathematiciens*, Nice France 1970, **1** (1971) 425, Gauthier-Villars, France; *Theorié de Hodge II* (in French), *Publ. Math. IHES* **40** (1971) 5; *Theorié de Hodge III* (in French), *Publ. Math. IHES* **44** (1971) 5;
J. Carlson, M. Green, P. Griffiths and J. Harris, *Infinitesimal variations of Hodge structure (I)*, *Compositio Math.* **50** (1983) 109.

[31] P. Griffiths, *On the periods of certain rational integrals. I*, *Annals Math.* **90** (1969) 460; *On the periods of certain rational integrals. II*, *Annals Math.* **90** (1969) 496; *A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles*, *Am. J. Math.* **101** (1979) 96.

[32] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658 [hep-th/9706109] [SPIRES].

[33] S. Kachru, S.H. Katz, A.E. Lawrence and J. McGreevy, *Open string instantons and superpotentials*, *Phys. Rev.* **D 62** (2000) 026001 [hep-th/9912151] [SPIRES]; *Mirror symmetry for open strings*, *Phys. Rev.* **D 62** (2000) 126005 [hep-th/0006047] [SPIRES].

[34] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, *Commun. Math. Phys.* **167** (1995) 301 [hep-th/9308122] [SPIRES].

[35] I. Gel'fand, M. Kapranov and A. Zelevinsky, *Hypergeometric functions and toric varieties*, *Funct. Anal. Appl.* **23** (1989) 12.

[36] P. Mayr, *$N = 1$ mirror symmetry and open/closed string duality*, *Adv. Theor. Math. Phys.* **5** (2002) 213 [hep-th/0108229] [SPIRES].

[37] M. Baumgartl, I. Brunner and M.R. Gaberdiel, *D-brane superpotentials and RG flows on the quintic*, *JHEP* **07** (2007) 061 [arXiv:0704.2666] [SPIRES].

[38] J. Knapp and E. Scheidegger, *Matrix factorizations, massey products and F-terms for two-parameter Calabi-Yau hypersurfaces*, arXiv:0812.2429 [SPIRES].

[39] M. Aganagic, A. Klemm and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, *Z. Naturforsch.* **A 57** (2002) 1 [hep-th/0105045] [SPIRES].

[40] *Puntos*, http://www.math.ucdavis.edu/∼deloera/RECENT_WORK/puntos2000;
Schubert, http://math.uib.no/schubert;
Instanton, http://www.math.uiuc.edu/∼katz/software.html.

[41] A. Klemm and S. Theisen, *Considerations of one modulus Calabi-Yau compactifications: Picard-Fuchs equations, Kähler potentials and mirror maps*, *Nucl. Phys.* **B 389** (1993) 153 [hep-th/9205041] [SPIRES].